

The exterior splash in $\text{PG}(6, q)$: Special conics

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October 17, 2014

Abstract

Let π be an order- q -subplane of $\text{PG}(2, q^3)$ that is exterior to ℓ_∞ . The exterior splash of π is the set of $q^2 + q + 1$ points on ℓ_∞ that lie on an extended line of π . Exterior splashes are projectively equivalent to scattered linear sets of rank 3, covers of the circle geometry $CG(3, q)$, and hyper-reguli of $\text{PG}(5, q)$. In this article we use the Bruck-Bose representation in $\text{PG}(6, q)$ to give a geometric characterisation of special conics of π in terms of the covers of the exterior splash of π . We also investigate properties of order- q -subplanes with a common exterior splash, and study the intersection of two exterior splashes.

1 Introduction

Let π be a subplane of $\text{PG}(2, q^3)$ of order q that is exterior to ℓ_∞ . We call π an *exterior order- q -subplane* of $\text{PG}(2, q^3)$. The lines of π meet ℓ_∞ in a set \mathbb{S} of size $q^2 + q + 1$, called the *exterior splash* of π on ℓ_∞ . The exterior splash turns out to be a set rich in geometric structure. In particular, [7] showed that the sets of points in an exterior splash has arisen in many different situations, namely scattered linear sets of rank 3, Sherk surfaces of size $q^2 + q + 1$, covers of the circle geometry $CG(3, q)$, and so hyper-reguli in $\text{PG}(5, q)$. Properties of the exterior splash in the Bruck-Bose representation in $\text{PG}(6, q)$ are studied in [8]. This current article continues this study, and furthers the investigation into the interplay between an exterior order- q -subplane in $\text{PG}(2, q^3)$, and its associated exterior splash.

This article proceeds as follows. In Section 2 we introduce the notation we use for the Bruck-Bose representation of $\text{PG}(2, q^3)$ in $\text{PG}(6, q)$, and give the background on exterior splashes needed to understand the results of this article. Section 3 looks at the Bruck-Bose representation of a class of special conics in an exterior order- q -subplane of $\text{PG}(2, q^3)$. In particular, Section 3.2 contains the main characterisation of this article. We show that the (π, ℓ_∞) -special conics in an exterior order- q -subplane π of $\text{PG}(2, q^3)$ correspond exactly

to the \mathbb{X} -special twisted cubics in $\text{PG}(6, q)$, where \mathbb{X} is a cover of an exterior splash. We then explore some of the fundamental geometric differences between the two covers of an exterior splash with respect to an associated order- q -subplane. Section 4 investigates order- q -subplanes with a common splash, and Section 5 looks at the intersection of two exterior splashes.

2 Background results

2.1 The Bruck-Bose representation of $\text{PG}(2, q^3)$ in $\text{PG}(6, q)$

Here we introduce the notation we will use for the Bruck-Bose representation of $\text{PG}(2, q^3)$ in $\text{PG}(6, q)$. We work with the finite field $\mathbb{F}_q = \text{GF}(q)$, and let $\mathbb{F}'_q = \mathbb{F}_q \setminus \{0\}$. A *2-spread* of $\text{PG}(5, q)$ is a set of $q^3 + 1$ planes that partition $\text{PG}(5, q)$. A *2-regulus* of $\text{PG}(5, q)$ is a set of $q + 1$ mutually disjoint planes π_1, \dots, π_{q+1} with the property that if a line meets three of the planes, then it meets all $q + 1$ of them. A 2-regulus \mathcal{R} has a set of $q^2 + q + 1$ mutually disjoint *ruling lines* that meet every plane of \mathcal{R} . A 2-spread \mathcal{S} is *regular* if for any three planes in \mathcal{S} , the 2-regulus containing them is contained in \mathcal{S} . See [18] for more information on 2-spreads.

The following construction of a regular 2-spread of $\text{PG}(5, q)$ will be needed. Embed $\text{PG}(5, q)$ in $\text{PG}(5, q^3)$ and let g be a line of $\text{PG}(5, q^3)$ disjoint from $\text{PG}(5, q)$. Let g^q, g^{q^2} be the conjugate lines of g ; both of these are disjoint from $\text{PG}(5, q)$. Let P_i be a point on g ; then the plane $\langle P_i, P_i^q, P_i^{q^2} \rangle$ meets $\text{PG}(5, q)$ in a plane. As P_i ranges over all the points of g , we get $q^3 + 1$ planes of $\text{PG}(5, q)$ that partition $\text{PG}(5, q)$. These planes form a regular 2-spread \mathcal{S} of $\text{PG}(5, q)$. The lines g, g^q, g^{q^2} are called the (conjugate skew) *transversal lines* of the 2-spread \mathcal{S} . Conversely, given a regular 2-spread in $\text{PG}(5, q)$, there is a unique set of three (conjugate skew) transversal lines in $\text{PG}(5, q^3)$ that generate \mathcal{S} in this way.

We will use the linear representation of a finite translation plane \mathcal{P} of dimension at most three over its kernel, due independently to André [1] and Bruck and Bose [13, 14]. Let Σ_∞ be a hyperplane of $\text{PG}(6, q)$ and let \mathcal{S} be a 2-spread of Σ_∞ . We use the phrase *a subspace of $\text{PG}(6, q) \setminus \Sigma_\infty$* to mean a subspace of $\text{PG}(6, q)$ that is not contained in Σ_∞ . Consider the following incidence structure: the *points* of $\mathcal{A}(\mathcal{S})$ are the points of $\text{PG}(6, q) \setminus \Sigma_\infty$; the *lines* of $\mathcal{A}(\mathcal{S})$ are the 3-spaces of $\text{PG}(6, q) \setminus \Sigma_\infty$ that contain an element of \mathcal{S} ; and *incidence* in $\mathcal{A}(\mathcal{S})$ is induced by incidence in $\text{PG}(6, q)$. Then the incidence structure $\mathcal{A}(\mathcal{S})$ is an affine plane of order q^3 . We can complete $\mathcal{A}(\mathcal{S})$ to a projective plane $\mathcal{P}(\mathcal{S})$; the points on the line at infinity ℓ_∞ have a natural correspondence to the elements of the 2-spread \mathcal{S} . The projective plane $\mathcal{P}(\mathcal{S})$ is the Desarguesian plane $\text{PG}(2, q^3)$ if and only if \mathcal{S} is a regular 2-spread of $\Sigma_\infty \cong \text{PG}(5, q)$ (see [10]). For the remainder of the article, we use \mathcal{S} to denote a regular 2-spread of $\Sigma_\infty \cong \text{PG}(5, q)$.

We use the following notation. If T is a point of ℓ_∞ in $\text{PG}(2, q^3)$, we use $[T]$ to refer to the plane of \mathcal{S} corresponding to T . More generally, if X is a set of points of $\text{PG}(2, q^3)$, then we let $[X]$ denote the corresponding set in $\text{PG}(6, q)$. If P is an affine point of $\text{PG}(2, q^3)$, we generally simplify the notation and also use P to refer to the corresponding affine point in $\text{PG}(6, q)$, although in some cases, to avoid confusion, we use $[P]$.

When \mathcal{S} is a regular 2-spread, we can relate the coordinates of $\mathcal{P}(\mathcal{S}) \cong \text{PG}(2, q^3)$ and $\text{PG}(6, q)$ as follows. Let τ be a primitive element in \mathbb{F}_{q^3} with primitive polynomial $x^3 - t_2x^2 - t_1x - t_0$. Then every element $\alpha \in \mathbb{F}_{q^3}$ can be uniquely written as $\alpha = a_0 + a_1\tau + a_2\tau^2$ with $a_0, a_1, a_2 \in \mathbb{F}_q$. Points in $\text{PG}(2, q^3)$ have homogeneous coordinates (x, y, z) with $x, y, z \in \mathbb{F}_{q^3}$. Let the line at infinity ℓ_∞ have equation $z = 0$; so the affine points of $\text{PG}(2, q^3)$ have coordinates $(x, y, 1)$. Points in $\text{PG}(6, q)$ have homogeneous coordinates $(x_0, x_1, x_2, y_0, y_1, y_2, z)$ with $x_0, x_1, x_2, y_0, y_1, y_2, z \in \mathbb{F}_q$. Let Σ_∞ have equation $z = 0$. Let $P = (\alpha, \beta, 1)$ be a point of $\text{PG}(2, q^3)$. We can write $\alpha = a_0 + a_1\tau + a_2\tau^2$ and $\beta = b_0 + b_1\tau + b_2\tau^2$ with $a_0, a_1, a_2, b_0, b_1, b_2 \in \mathbb{F}_q$. We want to map the element α of \mathbb{F}_{q^3} to the vector (a_0, a_1, a_2) , and we use the following notation to do this:

$$[\alpha] = (a_0, a_1, a_2).$$

This gives us some notation for the Bruck-Bose map, denoted ϵ , from an affine point $P = (\alpha, \beta, 1) \in \text{PG}(2, q^3) \setminus \ell_\infty$ to the corresponding affine point $[P] \in \text{PG}(6, q) \setminus \Sigma_\infty$, namely $\epsilon(\alpha, \beta, 1) = [(\alpha, \beta, 1)] = ([\alpha], [\beta], 1) = (a_0, a_1, a_2, b_0, b_1, b_2, 1)$. More generally, if $z \in \mathbb{F}_q$, then $\epsilon(\alpha, \beta, z) = ([\alpha], [\beta], z) = (a_0, a_1, a_2, b_0, b_1, b_2, z)$.

Consider the case when $z = 0$, a point on ℓ_∞ in $\text{PG}(2, q^3)$ has coordinates $L = (\alpha, \beta, 0)$ for some $\alpha, \beta \in \mathbb{F}_{q^3}$. In $\text{PG}(6, q)$, the point $\epsilon(\alpha, \beta, 0) = ([\alpha], [\beta], 0)$ is one point in the spread element $[L]$ corresponding to L . Moreover, the spread element $[L]$ consists of all the points $\{([\alpha x], [\beta x], 0) : x \in \mathbb{F}_{q^3}'\}$.

2.2 Introducing exterior splashes

This article relies heavily on properties of exterior splashes proved in [7, 8]. In this section we detail the most important definitions and results that are needed in this article. We begin with the important result that all exterior splashes are projectively equivalent.

Theorem 2.1 [7] *Consider the collineation group $G = \text{PGL}(3, q^3)$ acting on $\text{PG}(2, q^3)$. The subgroup G_ℓ fixing a line ℓ is transitive on the order- q -subplanes that are exterior to ℓ , and is transitive on the exterior splashes of ℓ .*

This theorem says that if we want to prove a result about exterior order- q -subplanes, or about exterior splashes, then we can without loss of generality prove it for a particular order- q -subplane. In [8], an order- q -subplane \mathcal{B} which is exterior to ℓ_∞ is coordinatised,

and \mathcal{B} is used in many of the proofs in this article. The main points of this coordinatisation are as follows.

Theorem 2.2 [8] *Let σ be the homography of $\text{PG}(2, q^3)$ with matrix*

$$K = \begin{pmatrix} -\tau & 1 & 0 \\ -\tau^q & 1 & 0 \\ \tau\tau^q & -\tau - \tau^q & 1 \end{pmatrix}.$$

Then σ maps the order- q -subplane $\pi_0 = \text{PG}(2, q)$ with exterior line $\ell = [-\tau\tau^q, \tau^q + \tau, -1]$ to an order- q -subplane $\mathcal{B} = \sigma(\pi_0)$ with exterior line $\ell_\infty = \sigma(\ell) = [0, 0, 1]$. Further, \mathcal{B} has exterior splash $\mathbb{S} = \{(k, 1, 0) : k \in \mathbb{F}_{q^3}, k^{q^2+q+1} = 1\} \equiv \{(\tau^{(q-1)^i}, 1, 0) : 0 \leq i < q^2 + q + 1\}$ and carriers $E_1 = (1, 0, 0)$ and $E_2 = (0, 1, 0)$.

We will also need the following result about exterior order- q -subplanes which have a common exterior splash and a common order- q -subline.

Theorem 2.3 [7] *In $\text{PG}(2, q^3)$, let \mathbb{S} be an exterior splash of ℓ_∞ , let ℓ be a line through a point of \mathbb{S} , and let b an order- q -subline of ℓ exterior to ℓ_∞ . Then there are exactly two order- q -subplanes that contain b and have exterior splash \mathbb{S} .*

We now consider an exterior splash \mathbb{S} of ℓ_∞ in $\text{PG}(2, q^3)$, and look at it in the Bruck-Bose representation in $\text{PG}(6, q)$. The set of $q^2 + q + 1$ points $\mathbb{S} \subset \ell_\infty$ corresponds to a set of $q^2 + q + 1$ planes (which we also denote by \mathbb{S}) of the regular 2-spread \mathcal{S} in $\Sigma_\infty \cong \text{PG}(5, q)$. As an exterior splash is equivalent to a cover of the circle geometry $CG(3, q)$, by [12], \mathbb{S} has two *switching sets* denoted \mathbb{X}, \mathbb{Y} , each containing $q^2 + q + 1$ planes of Σ_∞ . The three sets $\mathbb{S}, \mathbb{X}, \mathbb{Y}$ are called *hyper-reguli* in [21]: each set covers the same set of points; two planes in the same set are disjoint; while two planes from different sets meet in a unique point. Note that in [9], we show that the only planes which meet every plane of \mathbb{S} in a point are those in \mathbb{X} and \mathbb{Y} , so in particular, \mathbb{S} has exactly two covers. In this article we call \mathbb{X} and \mathbb{Y} *covers* of the exterior splash \mathbb{S} . If π is an exterior order- q -subplane of $\text{PG}(2, q^3)$ with exterior splash \mathbb{S} , then the two covers have different properties with respect to π . One cover is denoted \mathbb{T} and is called the *tangent cover* of π (or the tangent cover of \mathbb{S} with respect to π). The name follows from [8, Theorem 5.3] which shows that the planes of \mathbb{T} are related to the tangent planes of $[\pi]$. The other cover is denoted \mathbb{C} and is called the *conic cover* of π , it is related to certain conics of π , see Theorem 3.3. In Section 3 we discuss some of the fundamental differences in the geometry of the two covers in relation to the order- q -subplane π .

In [8], we show that in the cubic extension $\text{PG}(5, q^3)$ of $\Sigma_\infty \cong \text{PG}(5, q)$, each set $\mathbb{S}, \mathbb{T}, \mathbb{C}$ has a unique triple of conjugate *transversal lines*. That is, the lines $g_{\mathbb{S}}, g_{\mathbb{S}}^q, g_{\mathbb{S}}^{q^2}$ of $\text{PG}(5, q^3) \setminus \text{PG}(5, q)$ meet every extended plane of \mathbb{S} , and these are the only lines of $\text{PG}(5, q^3)$ which meet every

extended plane of \mathbb{S} (in fact, these are the three transversal lines of the regular 2-spread \mathcal{S}). Similarly, there are exactly three lines $g_{\mathbb{T}}, g_{\mathbb{T}}^q, g_{\mathbb{T}}^{q^2}$ of $\text{PG}(5, q^3) \setminus \text{PG}(5, q)$ that meet every extended plane of the cover \mathbb{T} ; and exactly three lines $g_{\mathbb{C}}, g_{\mathbb{C}}^q, g_{\mathbb{C}}^{q^2}$ of $\text{PG}(5, q^3) \setminus \text{PG}(5, q)$ that meet every extended plane of the cover \mathbb{C} .

We define the notion of \mathbb{X} -special conics/twisted cubics in $\text{PG}(6, q)$. In this article, this definition is used when \mathbb{X} is the regular 2-spread \mathcal{S} , or an exterior splash \mathbb{S} , or one of the covers \mathbb{T} or \mathbb{C} of an exterior splash.

- Definition 2.4**
1. An \mathbb{X} -special conic is a non-degenerate conic \mathcal{C} contained in a plane of \mathbb{X} , such that the extension of \mathcal{C} to $\text{PG}(6, q^3)$ meets the three transversal lines of \mathbb{X} .
 2. An \mathbb{X} -special twisted cubic is a twisted cubic \mathcal{N} in a 3-space of $\text{PG}(6, q) \setminus \Sigma_{\infty}$ about a plane of \mathbb{X} , such that the extension of \mathcal{N} to $\text{PG}(6, q^3)$ meets the three transversal lines of \mathbb{X} .

Note that an \mathbb{X} -special twisted cubic has no points in Σ_{∞} . Finally, we need the following result from [4] which describes the representation of certain order- q -sublines of $\text{PG}(2, q^3)$ in the Bruck-Bose representation in $\text{PG}(6, q)$.

- Theorem 2.5**
1. Order- q -sublines contained in ℓ_{∞} in $\text{PG}(2, q^3)$ correspond exactly to 2-reguli of \mathcal{S} in $\text{PG}(6, q)$.
 2. Order- q -sublines exterior to ℓ_{∞} in $\text{PG}(2, q^3)$ correspond exactly to \mathcal{S} -special twisted cubics in $\text{PG}(6, q)$.

3 Special conics of an exterior subplane

The main result of this section is to show that a certain type of conic in an exterior order- q -subplane in $\text{PG}(2, q^3)$ corresponds exactly to \mathbb{C} -special twisted cubics in $\text{PG}(6, q)$. We first introduce the idea in $\text{PG}(2, q^3)$ of a special conic in an exterior order- q -subplane.

3.1 Special conics

We are interested in a particular class of conics in an exterior order- q -subplane in $\text{PG}(2, q^3)$ which have a nice geometric representation in $\text{PG}(6, q)$. We define these special conics as follows. Note first that we can generalise the notion of an exterior splash on ℓ_{∞} to define the exterior splash \mathbb{S} of an order- q -subplane π onto any exterior line ℓ : \mathbb{S} is the set of

$q^2 + q + 1$ points on ℓ that lie on an extended line of π . An important collineation group acting on $\text{PG}(2, q^3)$ is $I = \text{PGL}(3, q^3)_{\pi, \ell}$ which fixes an order- q -subplane π , and a line ℓ exterior to π . By [7, Theorem 2.2], I fixes exactly three lines, namely ℓ and its conjugates m, n with respect to π . Further I fixes exactly three points: $E_1 = \ell \cap m$, $E_2 = \ell \cap n$, $E_3 = m \cap n$, which are conjugate with respect to π . Note that if $\pi_0 = \text{PG}(2, q)$, then $I = \text{PGL}(3, q^3)_{\pi_0, \ell}$ fixes the three exterior lines ℓ, ℓ^q, ℓ^{q^2} and the three points $E = \ell \cap \ell^{q^2}$, E^q, E^{q^2} .

The group $I = \text{PGL}(3, q^3)_{\pi, \ell}$ identifies two special points $E_1 = \ell \cap m$, $E_2 = \ell \cap n$ on ℓ which are called the *carriers* of the exterior splash \mathbb{S} of π onto ℓ . This is consistent with the definition of carriers of a circle geometry $CG(3, q)$, see [7, Theorem 4.2]. The fixed points and fixed lines of I are used to define an important class of conics in π .

Definition 3.1 *A (π, ℓ) -special conic of an order- q -subplane π (with exterior line ℓ) is a conic of π whose extension to $\text{PG}(2, q^3)$ contains the three fixed points E_1, E_2, E_3 of $I = \text{PGL}(3, q^3)_{\pi, \ell}$.*

It is straightforward to show that a (π, ℓ) -special conic is irreducible. We note that the incidence structure with *points* the points of an exterior order- q -subplane π , *lines* the (π, ℓ_∞) -special conics of π , and natural incidence is isomorphic to $\text{PG}(2, q)$. This observation is useful when counting special conics in this article. We also note that the set of $q^2 + q + 1$ (π, ℓ_∞) -special conics of π form a circumscribed bundle of conics in the sense of [3].

In the case when q is even, the set of special conics in an exterior order- q -subplane has some interesting properties relating to the nuclei. We look at the structure of the set of (π, ℓ_∞) -special conics through a fixed point P in π .

Theorem 3.2 *Let π be an exterior order- q -subplane of $\text{PG}(2, q^3)$, q even. Let P be a point of π , let $\mathcal{C}_0, \dots, \mathcal{C}_q$ be the (π, ℓ_∞) -special conics of π through P , and let N_k be the nucleus of \mathcal{C}_k . Then the points N_0, \dots, N_q are distinct, and lie on a line n_P not through P , called the nucleus line of P . Further, every line of π is the nucleus line for a distinct point of π .*

Proof By Theorem 2.1 and [7, Theorem 2.2], we can without loss of generality prove this for the order- q -subplane $\pi_0 = \text{PG}(2, q)$ exterior to the line $\ell = [-\tau\tau^q, \tau^q + \tau, -1]$, and the point $P = (0, 0, 1) \in \pi_0$. The fixed lines of $I = \text{PGL}(3, q^3)_{\pi_0, \ell}$ are ℓ, ℓ^q, ℓ^{q^2} , and the fixed points of I are $E = \ell \cap \ell^{q^2} = (1, \tau, \tau^2)$, $E^q = \ell^q \cap \ell$ and $E^{q^2} = \ell^{q^2} \cap \ell$. The conics $\mathcal{C}_0: y^2 - xz = 0$ and $\mathcal{C}_\infty: -t_0x^2 + yz - t_2xz - t_1xy = 0$ are conics of π_0 which contain the four points P, E, E^q and E^{q^2} , so they are (π_0, ℓ) -special conics containing P . Hence the conics in the pencil $\{\mathcal{C}_k = \mathcal{C}_0 + k\mathcal{C}_\infty : k \in \mathbb{F}_q \cup \{\infty\}\}$ are the $q + 1$ (π_0, ℓ) -special conics of π_0 containing P . The nucleus of conic \mathcal{C}_k is $N_k = (k, 1 - kt_2, -kt_1)$, $k \in \mathbb{F}_q \cup \{\infty\}$. So the

nuclei are distinct for distinct k , and all lie on the line $n_P = [t_1, 0, 1]$, a line not through P .

Let P, Q be two points of π_0 with nucleus lines n_P, n_Q respectively. We show that n_P, n_Q are distinct. Let $N \in n_P \cap n_Q$. There is a unique (π_0, ℓ) -special conic \mathcal{C} with nucleus N . As $N \in n_P$, \mathcal{C} contains P , and as $N \in n_Q$, \mathcal{C} contains Q . So \mathcal{C} is the unique (π_0, ℓ) -special conic through P and Q . Hence there can only be one point in the intersection $n_P \cap n_Q$, so the lines n_P, n_Q are distinct. Hence each line of π_0 is the nucleus line for a distinct point of π_0 . \square

A consequence of this theorem is that there is a bijection between lines of an order- q -subplane π and (π, ℓ) -special conics of π as follows. Firstly, there is a bijection mapping a point $P \in \pi$ to the unique (π, ℓ) -special conic of π with nucleus P . Then Theorem 3.2 gives a bijection mapping a point $P \in \pi$ to its nucleus line n_P .

3.2 Special conics are special twisted cubics

Let \mathbb{S} be an exterior splash of $\text{PG}(6, q)$ with covers \mathbb{C} and \mathbb{T} , so \mathbb{S} is contained in the regular 2-spread \mathcal{S} . In Definition 2.4 we defined the notion of \mathcal{S} -, \mathbb{S} -, \mathbb{C} - and \mathbb{T} -special twisted cubics. We consider the representation of each of these special twisted cubics in $\text{PG}(2, q^3)$.

We can characterise \mathcal{S} -special and \mathbb{S} -special twisted cubics as follows. By Theorem 2.5, a twisted cubic \mathcal{N} of $\text{PG}(6, q)$ corresponds to an exterior order- q -subline of $\text{PG}(2, q^3)$ if and only if \mathcal{N} is an \mathcal{S} -special twisted cubic. Hence if π is an exterior order- q -subplane of $\text{PG}(2, q^3)$ with exterior splash \mathbb{S} , then the order- q -sublines of π are precisely the \mathbb{S} -special twisted cubics in $\text{PG}(6, q)$.

In this section, we characterise \mathbb{C} - and \mathbb{T} -special twisted cubics, and show they correspond precisely to special conics of some exterior order- q -subplane of $\text{PG}(2, q^3)$. The argument proving this proceeds as follows. In Theorem 3.3, we show that a (π, ℓ_∞) -special conic \mathcal{C} in an exterior order- q -subplane π of $\text{PG}(2, q^3)$ corresponds to a twisted cubic $[\mathcal{C}]$ in $\text{PG}(6, q)$. Moreover, the 3-space containing $[\mathcal{C}]$ contains a unique plane of the cover \mathbb{C} (this is why we named \mathbb{C} the ‘conic cover’). Theorem 3.5 shows that the twisted cubic $[\mathcal{C}]$ is \mathbb{C} -special. Theorem 3.6 proves the converse, that each \mathbb{C} -special twisted cubic of $\text{PG}(6, q)$ corresponds to a (π', ℓ_∞) -special conic in some exterior order- q -subplane π' of $\text{PG}(2, q^3)$. Note also that Corollary 3.8 shows that a non-special conic of π corresponds to a 6-dimensional normal rational curve of $\text{PG}(6, q)$. Finally, Corollary 3.9 shows that every \mathbb{T} -special twisted cubic meets π in at most three points, but corresponds to a special conic in some other exterior order- q -subplane of $\text{PG}(2, q^3)$.

Theorem 3.3 *Let \mathcal{C} be a (π, ℓ_∞) -special conic in an exterior order- q -subplane π of $\text{PG}(2, q^3)$. Then in $\text{PG}(6, q)$, \mathcal{C} corresponds to a twisted cubic $[\mathcal{C}]$, and the 3-space containing $[\mathcal{C}]$ meets*

Σ_∞ in a plane of the conic cover \mathbb{C} of π .

Proof By Theorem 2.1, we can without loss of generality prove this for the order- q -subplane \mathcal{B} coordinatised in Theorem 2.2. Further, by [7, Theorem 6.1], we can without loss of generality prove it for any $(\mathcal{B}, \ell_\infty)$ -special conic in \mathcal{B} . In $\text{PG}(2, q^3)$, consider the order- q -subplane $\pi_0 = \text{PG}(2, q)$ with exterior line $\ell = [-\tau\tau^q, \tau^q + \tau, -1]$, and let \mathcal{C}' be the conic in π_0 of equation $y^2 = zx$. By [7, Lemma 2.4], the exterior splash of π_0 onto ℓ has carriers $E = (1, \tau, \tau^2)$ and E^q . These both lie on the extension of \mathcal{C}' to $\text{PG}(2, q^3)$, as does E^{q^2} , hence \mathcal{C}' is a (π_0, ℓ) -special conic of π_0 . By Theorem 2.2, the homography σ with matrix K maps π_0, ℓ to \mathcal{B}, ℓ_∞ respectively. Further, $\mathcal{C} = \sigma(\mathcal{C}')$ is a $(\mathcal{B}, \ell_\infty)$ -special conic in \mathcal{B} . The points of \mathcal{C}' are $(1, \theta, \theta^2)$, $\theta \in \mathbb{F}_q \cup \{\infty\}$, so \mathcal{C} has points $P_\theta = K(1, \theta, \theta^2)^t$, $\theta \in \mathbb{F}_q \cup \{\infty\}$. Now $P_\infty = (0, 0, 1)$, and for $\theta \in \mathbb{F}_q$, we have

$$\begin{aligned} P_\theta &= (-\tau + \theta, -\tau^q + \theta, \tau\tau^q - \theta(\tau^q + \tau) + \theta^2) \\ &\equiv (-(\theta - \tau)(\theta - \tau^{q^2}), -(\theta - \tau^q)(\theta - \tau^{q^2}), (\theta - \tau)(\theta - \tau^q)(\theta - \tau^{q^2})). \end{aligned}$$

Note that the first two coordinates are polynomials in θ of degree 2, and the third coordinate is a polynomial in θ of degree 3. We write $P_\theta = (G(\theta)^q, G(\theta), f(\theta))$ where $G(\theta) = -(\theta - \tau^q)(\theta - \tau^{q^2})$ is a polynomial over \mathbb{F}_{q^3} and $f(\theta) = (\theta - \tau)^{q^2+q+1}$ is a polynomial over \mathbb{F}_q . For $\theta \in \mathbb{F}_q$, the third coordinate $f(\theta)$ is an element of \mathbb{F}_q , hence the corresponding point in $\text{PG}(6, q)$ is $P_\theta = ([G(\theta)^q], [G(\theta)], f(\theta))$. As each coordinate of P_θ is a polynomial in θ of degree at most three, and the coordinates have no common factors over \mathbb{F}_q , the set of points $[\mathcal{C}] = \{P_\theta : \theta \in \mathbb{F}_q\} \cup \{P_\infty = (0, 0, 1)\}$ is a rational curve of order three in $\text{PG}(6, q)$.

We show that $[\mathcal{C}]$ is normal in a 3-space by considering the projection of $[\mathcal{C}]$ onto Σ_∞ from the point P_∞ . Straightforward calculations show that the point $Q_\theta = P_\infty P_\theta \cap \Sigma_\infty$ has coordinates $Q_\theta = ([G(\theta)^q], [(G(\theta)^q)^{q^2}], 0)$. By [8, Lemma 5.1], one of the planes of the conic cover \mathbb{C} of \mathcal{B} is $[C_1] = \{([x], [x^{q^2}], 0) : x \in \mathbb{F}_{q^3}\}$. So the point Q_θ lies in the cover plane $[C_1]$ for all $\theta \in \mathbb{F}_q$. We use the identities $\tau^q + \tau^{q^2} = t_2 - \tau$ and $\tau^q \tau^{q^2} = \tau^2 - t_2 \tau - t_1$ to simplify $G(\theta)$ to $G(\theta) = (-\theta^2 + t_2 \theta + t_1) + (-\theta + t_2)\tau - \tau^2$. So $[G(\theta)] = \theta^2(-1, 0, 0) + \theta(t_2, -1, 0) + (t_1, t_2, -1)$. Hence the points Q_θ , $\theta \in \mathbb{F}_q$, lie on an irreducible conic in $[C_1]$. Hence the points of $[\mathcal{C}]$ span the 3-space $\langle P_\infty, [C_1] \rangle$, and so the conic \mathcal{C} corresponds to a twisted cubic $[\mathcal{C}]$ in a 3-space that meets Σ_∞ in a plane of \mathbb{C} . \square

Corollary 3.4 *Let π be an exterior order- q -subplane of $\text{PG}(2, q^3)$ with conic cover \mathbb{C} . Distinct (π, ℓ_∞) -special conics in π correspond to twisted cubics lying in 3-spaces through distinct planes of \mathbb{C} .*

Proof By [7, Theorem 6.1], the singer cycle $I = \text{PGL}(3, q^3)_{\mathcal{B}, \ell_\infty}$ acts regularly on the $(\mathcal{B}, \ell_\infty)$ -special conics of \mathcal{B} . Further, by [8, Lemma 5.2], in $\text{PG}(6, q)$, $[I]$ is a singer cycle that acts regularly on the cover planes of \mathbb{C} . There are $q^2 + q + 1$ planes in \mathbb{C} , and by

[7], there are $q^2 + q + 1$ (π, ℓ_∞) -special conics in π . Hence every (π, ℓ_∞) -special conic corresponds to a twisted cubic that lies in a 3-space about a distinct plane of the conic cover \mathbb{C} . \square

We now show that the twisted cubics of $\text{PG}(6, q)$ corresponding to (π, ℓ) -special conics of π in $\text{PG}(2, q^3)$ are \mathbb{C} -special twisted cubics. That is, in the cubic extension $\text{PG}(6, q^3)$, they meet the transversals $g_{\mathbb{C}}, g_{\mathbb{C}}^q, g_{\mathbb{C}}^{q^2}$ of the conic cover \mathbb{C} of π .

Theorem 3.5 *Let π be an exterior order- q -subplane of $\text{PG}(2, q^3)$ with conic cover \mathbb{C} . A (π, ℓ_∞) -special conic of π corresponds in $\text{PG}(6, q)$ to a \mathbb{C} -special twisted cubic.*

Proof As in the proof of Theorem 3.3, we can without loss of generality prove this for the order- q -subplane \mathcal{B} coordinatised in Theorem 2.2, and the $(\mathcal{B}, \ell_\infty)$ -special conic $\mathcal{C} = \sigma(\mathcal{C}')$ in \mathcal{B} . We continue our calculations from the end of proof of Theorem 3.3, using the same notation. That is, let $G(\theta) = -(\theta - \tau^q)(\theta - \tau^{q^2})$, $f(\theta) = (\theta - \tau)(\theta - \tau^q)(\theta - \tau^{q^2})$ and $P_\theta = ([G(\theta)^q], [G(\theta)], f(\theta))$. Then the set $[\mathcal{C}] = \{P_\theta : \theta \in \mathbb{F}_q \cup \{\infty\}\}$ is a twisted cubic which we denote by $\mathcal{N} = [\mathcal{C}]$. We can uniquely write $G(\theta) = g_0(\theta) + g_1(\theta)\tau + g_2(\theta)\tau^2$ and $G(\theta)^q = h_0(\theta) + h_1(\theta)\tau + h_2(\theta)\tau^2$ where $g_i(\theta), h_i(\theta), i = 0, 1, 2$, are polynomials over \mathbb{F}_q . As $f(\theta)$ is a polynomial over \mathbb{F}_q , the extension \mathcal{N}^* of \mathcal{N} to $\text{PG}(6, q^3)$ has points $P_\theta = (h_0(\theta), h_1(\theta), h_2(\theta), g_0(\theta), g_1(\theta), g_2(\theta), f(\theta))$, for $\theta \in \mathbb{F}_{q^3} \cup \{\infty\}$. As $f(\theta)$ has zeros τ, τ^q, τ^{q^2} , \mathcal{N}^* meets Σ_∞ in the points P_τ, P_{τ^q} and $P_{\tau^{q^2}}$.

By the proof of Theorem 3.3, the 3-space containing \mathcal{N} meets Σ_∞ in the cover plane $[C_1]$ of \mathbb{C} . We recall [8, Theorem 6.3] which shows that the transversal line $g_{\mathbb{C}} = \langle A_1, A_2^q \rangle$ of \mathbb{C} meets the (extended) cover plane $[C_1]$ in the point $A_1 + \eta^{1-q}A_2^q$, where $A_1 = (p_0, p_1, p_2, 0, 0, 0, 0)$, $A_2 = (0, 0, 0, p_0, p_1, p_2, 0)$, $p_0 = t_1 + t_2\tau - \tau^2$, $p_1 = t_2 - \tau$, $p_2 = -1$ and $\eta = p_0 + p_1\tau + p_2\tau^2$. We will show that the point P_{τ^q} lies on the transversal $g_{\mathbb{C}}$. As in the proof of Theorem 3.3, on simplifying, we have $G(\theta) = (-\theta^2 + t_2\theta + t_1) + (-\theta + t_2)\tau - \tau^2$, so $g_0(\theta) = -\theta^2 + t_2\theta + t_1$, $g_1(\theta) = -\theta + t_2$ and $g_2(\theta) = -1$. Letting $A = (p_0, p_1, p_2)$, and noting that $g_0(\tau^q) = p_0^q$, $g_1(\tau^q) = p_1^q$, $g_2(\tau^q) = p_2^q$, we have $A_2^q = (0, 0, 0, g_0(\tau^q), g_1(\tau^q), g_2(\tau^q), 0) = ([0], A^q, 0)$. Thus $P_\tau = ([A^q]^q, A^q, 0)$. It follows from [8, Equation (8)] that $(A^q)^q = \eta^{q-1}A$. Hence $P_\tau = (\eta^{q-1}A, A^q, 0) = \eta^{q-1}A_1 + A_2^q \equiv A_1 + \eta^{1-q}A_2^q$. Hence $P_{\tau^q} = g_{\mathbb{C}} \cap [C_1]^* = g_{\mathbb{C}} \cap \mathcal{N}^*$. Similarly, \mathcal{N}^* meets the transversals $g_{\mathbb{C}}^q, g_{\mathbb{C}}^{q^2}$, and so $\mathcal{N} = [\mathcal{C}]$ is a \mathbb{C} -special twisted cubic. \square

The converse of this result also holds. That is, a \mathbb{C} -special twisted cubic corresponds to a special conic in some exterior order- q -subplane with conic cover \mathbb{C} .

Theorem 3.6 *Let \mathbb{S} be an exterior splash in $\text{PG}(6, q)$, $q \geq 3$, let \mathbb{X} be one cover of \mathbb{S} , and let $[\mathcal{N}]$ be an \mathbb{X} -special twisted cubic. Then in $\text{PG}(2, q^3)$, there is a unique exterior order- q -subplane π that contains \mathcal{N} . Further, π has exterior splash \mathbb{S} , conic cover \mathbb{X} , and \mathcal{N} is a (π, ℓ_∞) -special conic of π .*

Proof Fix an exterior splash \mathbb{S} . Let π be any exterior order- q -subplane of $\text{PG}(2, q^3)$ with exterior splash \mathbb{S} and conic cover \mathbb{C} , and let \mathcal{C} be a (π, ℓ_∞) -special conic in π . By Theorem 3.5, in $\text{PG}(6, q)$, \mathcal{C} corresponds to a \mathbb{C} -special twisted cubic. We show that the converse is true by counting the two sets and showing they have the same size. Let x be the number of pairs (\mathcal{C}, π) where π is an exterior order- q -subplane of $\text{PG}(2, q^3)$ with the given exterior splash \mathbb{S} , and \mathcal{C} is a (π, ℓ_∞) -special conic of π . In $\text{PG}(6, q)$, let \mathbb{X} be a cover of the given exterior splash \mathbb{S} , and let y be the number of \mathbb{X} -special twisted cubics. Note that for any set \mathcal{N} of $q + 1$ affine points of $\text{PG}(2, q^3)$, if $[\mathcal{N}]$ is an \mathbb{X} -special twisted cubic of $\text{PG}(6, q)$, then \mathcal{N} contains a quadrangle, so \mathcal{N} lies in at most one exterior order- q -subplane. Further, if \mathcal{N} does lie in an order- q -subplane with exterior splash \mathbb{S} , then by Theorem 3.5, \mathcal{N} may be a (π, ℓ_∞) -special conic of π . That is, $x \geq y$ with equality if and only if every \mathbb{X} -special twisted cubic corresponds to a (π, ℓ_∞) -special conic in an exterior order- q -subplane π with exterior splash \mathbb{S} .

We first count x . By [7, Theorem 4.5], there are $2q^6(q^3 - 1)$ exterior order- q -subplanes with the fixed exterior splash \mathbb{S} . Further, an exterior order- q -subplane contains $q^2 + q + 1$ special conics. Hence $x = 2q^6(q^3 - 1)(q^2 + q + 1)$. Now we count y . Note that for each cover \mathbb{X} of \mathbb{S} , an \mathbb{X} -special twisted cubic lies in a 3-space that meets Σ_∞ in a plane α of the cover \mathbb{X} . There are $2(q^2 + q + 1)$ choices for a cover plane α , q^3 3-spaces of $\text{PG}(6, q) \setminus \Sigma_\infty$ through α , and by [5, Lemma 2.5] there are $q^3(q^3 - 1)$ \mathbb{X} -special twisted cubics in such a 3-space. Hence $y = 2(q^2 + q + 1)q^6(q^3 - 1)$. Thus $x = y$, hence each \mathbb{X} -special twisted cubic $[\mathcal{N}]$ (in a 3-space that meets Σ_∞ in a cover plane of \mathbb{X}) corresponds to a set of points \mathcal{N} in $\text{PG}(2, q^3)$ that lie in a unique exterior order- q -subplane π with exterior splash \mathbb{S} . Further, \mathcal{N} is a (π, ℓ_∞) -special conic in π , and so by Theorem 3.5, \mathbb{X} is the conic cover of π . \square

We now consider a non-special conic of the exterior order- q -subplane π , we will show that it corresponds to a 6-dimensional normal rational curve in $\text{PG}(6, q)$. First we show that all conics of π correspond to 6- or 3-dimensional normal rational curves of $\text{PG}(6, q)$.

Lemma 3.7 *Let \mathcal{C} be an irreducible conic of an exterior order- q -subplane π . In $\text{PG}(6, q)$, $[\mathcal{C}]$ is either a 6-dimensional normal rational curve or a twisted cubic.*

Proof By Theorem 2.1, we can without loss of generality prove this for the order- q -subplane \mathcal{B} coordinatised in Theorem 2.2. We first calculate the equation of a general irreducible conic \mathcal{C}' in $\pi_0 = \text{PG}(2, q)$, then use the homography σ from Theorem 2.2 to map it to a general irreducible conic \mathcal{C} in \mathcal{B} . Let \mathcal{C}' be the image of the conic $\mathcal{C}'' = \{(1, \theta, \theta^2) : \theta \in \mathbb{F}_q \cup \{\infty\}\} \subset \pi_0$ under a homography of $\text{PG}(2, q)$ with matrix $A = (a_{ij})$. The point $(1, \theta, \theta^2) \in \mathcal{C}''$ is mapped to the point $(f_0(\theta), f_1(\theta), f_2(\theta)) \in \mathcal{C}'$, for $\theta \in \mathbb{F}_q \cup \{\infty\}$, where $f_0(\theta) = a_{00} + a_{01}\theta + a_{02}\theta^2$, $f_1(\theta) = a_{10} + a_{11}\theta + a_{12}\theta^2$, $f_2(\theta) = a_{20} + a_{21}\theta + a_{22}\theta^2$, are polynomials over \mathbb{F}_q . Under the homography σ with matrix K of Theorem 2.2, this is mapped to a general irreducible conic \mathcal{C} of \mathcal{B} with points

$$F_\theta = (-f_0(\theta)\tau + f_1(\theta), -f_0(\theta)\tau^q + f_1(\theta), f_0(\theta)\tau\tau^q - f_1(\theta)(\tau + \tau^q) + f_2(\theta)), \quad (1)$$

$\theta \in \mathbb{F}_q \cup \{\infty\}$. We write this as $F_\theta = (g_0(\theta), g_1(\theta), g_2(\theta))$. Multiply all the coordinates of F_θ by $g_2(\theta)^{q^2+q}$, so $F_\theta \equiv (g_0(\theta)g_2(\theta)^{q^2+q}, g_1(\theta)g_2(\theta)^{q^2+q}, g_2(\theta)^{q^2+q+1})$, and now the last coordinate lies in \mathbb{F}_q . So in $\text{PG}(6, q)$,

$$[\mathcal{C}] = \{F_\theta \equiv ([g_0(\theta)g_2(\theta)^{q^2+q}], [g_1(\theta)g_2(\theta)^{q^2+q}], [g_2(\theta)^{q^2+q+1}]) : \theta \in \mathbb{F}_q \cup \{\infty\}\}.$$

Note that $g_0(\theta), g_1(\theta), g_2(\theta)$ have degree at most two in θ , and so $g_2(\theta)^{q^2+q} = g_2(\theta)^{q^2} g_2(\theta)^q$ has degree at most four in θ . Hence each coordinate of the points F_θ of $[\mathcal{C}]$ is a polynomial in θ of degree at most six. Thus $[\mathcal{C}]$ is a rational curve in $\text{PG}(6, q)$ of order at most 6. We want to show that $[\mathcal{C}]$ is normal in either 6-space, or in a 3-space.

We begin by showing that $g_2(\theta) = f_0(\theta)\tau\tau^q - f_1(\theta)(\tau + \tau^q) + f_2(\theta)$ has degree two in θ . Expanding gives $g_2(\theta) = (a_{00}\tau\tau^q - a_{10}(\tau + \tau^q) + a_{20}) + (a_{01}\tau\tau^q - a_{11}(\tau + \tau^q) + a_{21})\theta + (a_{02}\tau\tau^q - a_{12}(\tau + \tau^q) + a_{22})\theta^2$. If $g_2(\theta)$ is of degree less than two, then $a_{02}\tau\tau^q - a_{12}(\tau + \tau^q) + a_{22} = 0$. Hence $a_{02} = a_{12} = a_{22} = 0$ (since $1, \tau\tau^q, \tau + \tau^q$ are linearly independent over $\text{GF}(q)$) contradicting $|A| \neq 0$. Thus $g_2(\theta)$ is a polynomial of degree two in θ over \mathbb{F}_{q^3} .

Further, if $g_2(\theta)$ was reducible over \mathbb{F}_q , then let $k \in \mathbb{F}_q$ be a root. Then the point F_k lies in ℓ_∞ and lies in the conic \mathcal{C} of \mathcal{B} , contradicting \mathcal{B} being a subplane exterior to ℓ_∞ . Hence $g_2(\theta)$ is irreducible over \mathbb{F}_q .

Suppose that $g_2(\theta)$ is irreducible over \mathbb{F}_{q^3} . Then $g_0(\theta), g_1(\theta), g_2(\theta)$ have no common factors over \mathbb{F}_q . Hence the set of points $[\mathcal{C}]$ is a normal rational curve of order six in $\text{PG}(6, q)$, which is the first possibility in the theorem.

Now suppose $g_2(\theta)$ is reducible over \mathbb{F}_{q^3} , so $g_2(\theta)$ has roots $\delta, \eta \in \mathbb{F}_{q^3} \setminus \mathbb{F}_q$. So $g_2(\theta)^{q^2+q+1} = r(\theta) \times s(\theta)$ where $r(\theta)$ is an irreducible cubic over \mathbb{F}_q , with roots $\delta, \delta^q, \delta^{q^2}$ over \mathbb{F}_{q^3} , and $s(\theta)$ is an irreducible cubic over \mathbb{F}_q with roots η, η^q, η^{q^2} in \mathbb{F}_{q^3} . If neither $r(\theta)$ nor $s(\theta)$ is a factor of both $g_0(\theta)g_2(\theta)^q g_2(\theta)^{q^2}$ and $g_1(\theta)g_2(\theta)^q g_2(\theta)^{q^2}$, then each coordinate has degree six in θ , and $g_0(\theta), g_1(\theta)$ and $g_2(\theta)$ have no common factors over \mathbb{F}_q , and hence the set of points $[\mathcal{C}]$ is a normal rational curve of order six in $\text{PG}(6, q)$, again the first possibility in the statement of the theorem.

Otherwise suppose that $r(\theta)$ say is a factor of both $g_0(\theta)g_2(\theta)^q g_2(\theta)^{q^2}$ and $g_1(\theta)g_2(\theta)^q g_2(\theta)^{q^2}$. As $g_0(\theta)$ is a polynomial in θ of degree one or two over \mathbb{F}_q , the roots of $g_0(\theta)$ lie in either \mathbb{F}_q or $\text{GF}(q^2)$, and so cannot be a root of $r(\theta)$ or $s(\theta)$, as their roots lie in $\mathbb{F}_{q^3} \setminus \mathbb{F}_q$. So if $r(\theta)$ is an irreducible (over \mathbb{F}_q) factor of $g_0(\theta)g_2(\theta)^q g_2(\theta)^{q^2}$, it follows that $r(\theta)$ divides $g_2(\theta)^q g_2(\theta)^{q^2}$, which has two factors (over \mathbb{F}_{q^3}) of $r(\theta)$, and two factors (over \mathbb{F}_{q^3}) of $s(\theta)$. Hence either $\eta = \delta^q$ or δ^{q^2} , and so $r(\theta) = cs(\theta)$ for some $c \in \mathbb{F}'_{q^3}$. Note that if $\eta = \delta^q$ we obtain $g_2(\theta) = c(\theta - \delta)(\theta - \delta^q)$, and if $\eta = \delta^{q^2}$ we obtain $g_2(\theta) = c(\theta - \delta)(\theta - \delta^{q^2})$. However, $(c(\theta - \delta)(\theta - \delta^{q^2}))^q = c^q(\theta - \delta)(\theta - \delta^q)$ so without loss of generality, we may assume that $\eta = \delta^q$. So in $\text{PG}(2, q^3)$, we have $F_\theta \equiv (g_0(\theta)(\theta - \delta^q)r(\theta), g_1(\theta)(\theta - \delta^q)r(\theta), cr(\theta)^2) \equiv (g_0(\theta)(\theta - \delta^q), g_1(\theta)(\theta - \delta^q), cr(\theta))$.

Hence in $\text{PG}(6, q)$,

$$[\mathcal{C}] = \{F_\theta = ([g_0(\theta)(\theta - \delta^q)], [g_1(\theta)(\theta - \delta^q)], cr(\theta)) : \theta \in \mathbb{F}_q \cup \{\infty\}\}. \quad (2)$$

That is, the rational curve of order six reduces to a rational curve of order three, so it lies in some 3-space. If $[\mathcal{C}]$ is not normal in this 3-space, then the coordinates of F_θ in (2) have a common linear factor over \mathbb{F}_q , a contradiction as $r(\theta)$ is irreducible over \mathbb{F}_q . Hence $[\mathcal{C}]$ is a normal rational curve in a 3-space, that is, a twisted cubic. So we have the two possibilities in the statement of the theorem. \square

As an immediate consequence of the preceding results, we have the following characterisation of the conics of an exterior order- q -subplane of $\text{PG}(2, q^3)$.

Corollary 3.8 *Let π be an exterior order- q -subplane of $\text{PG}(2, q^3)$ with conic cover \mathbb{C} , and let \mathcal{C} be a conic of π . Then \mathcal{C} is a (π, ℓ_∞) -special conic of π if and only if in $\text{PG}(6, q)$, $[\mathcal{C}]$ is a \mathbb{C} -special twisted cubic. Otherwise, $[\mathcal{C}]$ is a 6-dimensional normal rational curve of $\text{PG}(6, q)$.*

Finally, we characterise \mathbb{T} -special conic of $\text{PG}(6, q)$ as follows. Let π be an exterior order- q -subplane of $\text{PG}(2, q^3)$ with tangent cover \mathbb{T} , and let $[\mathcal{N}]$ be a \mathbb{T} -special twisted cubic of $\text{PG}(6, q)$. By Theorem 3.6, there is a unique order- q -subplane π' of $\text{PG}(2, q^3)$ that contains the points of \mathcal{N} , and further π' has conic cover \mathbb{T} . That is, \mathcal{N} is a (π', ℓ_∞) -special conic of π' . Also, by [7, Theorem 7.2], π and π' meet in either an order- q -subline, or in at most three points. In either case, \mathcal{N} has at most three points in π . In summary, we have:

Corollary 3.9 *Let π be an exterior order- q -subplane in $\text{PG}(2, q^3)$ with exterior splash \mathbb{S} , conic cover \mathbb{C} and tangent cover \mathbb{T} . A \mathbb{T} -special twisted cubic of $\text{PG}(6, q)$ corresponds to a set \mathcal{C} of $q + 1$ points in $\text{PG}(2, q^3)$, at most three of which lie in π . Further, \mathcal{C} lies in a unique exterior order- q -subplane π' , and forms a (π', ℓ_∞) -special conic in π' .*

We note that comments in Section 3.4 further support this fundamental difference in behaviour between the two covers of \mathbb{S} with respect to an associated exterior order- q -subplane π .

3.3 Tangent lines to special conics in $\text{PG}(6, q)$

We continue our study of special conics in an exterior order- q -subplane of $\text{PG}(2, q^3)$ in this section by looking at a relationship between tangent lines of special conics in $\text{PG}(2, q^3)$ and tangent lines of special twisted cubics in $\text{PG}(6, q)$. It is straightforward to show that in an exterior order- q -subplane π , there is a unique (π, ℓ_∞) -special conic \mathcal{C} through a fixed point P tangent to a fixed line ℓ of π through P . We show that in $\text{PG}(6, q)$, the tangent lines to the two special twisted cubics $[\mathcal{C}]$ and $[\ell]$ at P are the same.

Theorem 3.10 *Let π be an exterior order- q -subplane, P a point of π , and ℓ a line of π through P . Let \mathcal{C} be the unique (π, ℓ_∞) -special conic of π through P with tangent ℓ . In $\text{PG}(6, q)$, the unique tangent at P to the \mathcal{S} -special twisted cubic $[\ell]$, and the unique tangent at P to the \mathbb{C} -special twisted cubic $[\mathcal{C}]$ are the same line.*

Proof By Theorem 2.1 and [7, Theorem 2.2], we can without loss of generality prove this for the exterior order- q -subplane \mathcal{B} coordinatised in Theorem 2.2, and the point $P = (0, 0, 1)$. We begin with the order- q -subplane $\pi_0 = \text{PG}(2, q)$ and the exterior line $\ell = [-\tau\tau^q, \tau^q + \tau, -1]$, and calculate the coordinates of the $q + 1$ (π_0, ℓ) -special conics \mathcal{C}_k ($k \in \mathbb{F}_q \cup \{\infty\}$) of π_0 through the point $P = (0, 0, 1)$. We then apply the homography σ from Theorem 2.2 to transform each conic \mathcal{C}_k to a $(\mathcal{B}, \ell_\infty)$ -special conic \mathcal{D}_k of \mathcal{B} through the point $\sigma(P) = P$.

By [7, Lemma 2.4], the exterior splash \mathbb{S} of π_0 on ℓ has carriers $E = (1, \tau, \tau^2)$, E^q . Consider the two conics $\mathcal{C}_0 : y^2 - xz = 0$ and $\mathcal{C}_\infty : -t_0x^2 + yz - t_2xz - t_1xy = 0$ of π_0 . They both contain the points P, E, E^q, E^{q^2} , hence each conic in the pencil $\{\mathcal{C}_k = \mathcal{C}_0 + k\mathcal{C}_\infty : k \in \mathbb{F}_q \cup \infty\}$ contains the four points P, E, E^q, E^{q^2} . That is, each \mathcal{C}_k is a (π_0, ℓ) -special conic of π_0 . For q both even and odd, the tangent line t'_k at P to \mathcal{C}_k is the line $[-1 - kt_2, k, 0] = [w, 1, 0]$ where

$$w = \frac{-kt_2 - 1}{k}. \quad (3)$$

Further, the conic \mathcal{C}_k consists of the points $C_{k,\theta} = (1, \theta, f(\theta))$ for $\theta \in \mathbb{F}_q \cup \{\infty\}$, where $f(\theta) = (-\theta^2 + kt_1\theta + kt_0)/(k\theta - kt_2 - 1)$. We now apply the homography σ of Theorem 2.2 with matrix K to map: the (π_0, ℓ) -special conic \mathcal{C}_k (with points $C_{k,\theta}$) of π_0 to a $(\mathcal{B}, \ell_\infty)$ -special conic \mathcal{D}_k (with points $D_{k,\theta}$) of \mathcal{B} ; and the tangent line t'_k to \mathcal{C}_k at P to the tangent line $t_k = \sigma(t'_k)$ to \mathcal{D}_k at P . In $\text{PG}(6, q)$, $[t_k]$ is an \mathbb{S} -special twisted cubic by Theorem 2.5, and $[\mathcal{D}_k]$ is a \mathbb{C} -special twisted cubic by Theorem 3.5 (where \mathbb{C} is the conic cover of \mathcal{B}).

Firstly, we consider the tangent line at the point P to the \mathbb{S} -special twisted cubic $[t_k]$, we denote the intersection of this tangent line with Σ_∞ by I_{P,t_k} . Note that points of π_0 on the line t'_k distinct from P have coordinates $P'_x = (1, -w, x)$ for $x \in \mathbb{F}_q$, so points of \mathcal{B} on the line t_k distinct from P have coordinates $P_x = \sigma(P'_x) = (-\tau - w, -\tau^q - w, \tau\tau^q + (\tau + \tau^q)w + x)$, for $x \in \mathbb{F}_q$. To convert this to a coordinate in $\text{PG}(6, q)$, we need to multiply by an element of \mathbb{F}_{q^3} so that the last coordinate lies in \mathbb{F}_q . Let $F(x) = \tau\tau^q + (\tau + \tau^q)m + x$ (the third coordinate in P_x). As $F(x) \in \mathbb{F}_{q^3}$, we have $F(x)^{q^2+q+1} \in \mathbb{F}_q$, so multiply the coordinates of P_x by $F(x)^{q^2+q}$. So the point $P_x \in \mathcal{B}$ corresponds in $\text{PG}(6, q)$ to the point P_x with coordinates $P_x = ([-(\tau + w)F(x)^{q^2+q}], [-(\tau^q + w)F(x)^{q^2+q}], F(x)^{q^2+q+1})$. To calculate I_{P,t_k} , let $Q_x = PP_x \cap \Sigma_\infty$. Now $F(x)^{q^2+q}$ is equal to x^2 plus lower powers of x . As $P = (0, 0, 1)$ is P_x with $x = \infty$, we can calculate $I_{P,t_k} = Q_\infty$ by dividing all the coordinates by x^2 and letting $x \rightarrow \infty$.

$$I_{P,t_k} = \lim_{x \rightarrow \infty} PP_x \cap \Sigma_\infty = ([-\tau - w], [-\tau^q - w], 0) \equiv w([1], [1], 0) + ([\tau], [\tau^q], 0).$$

Secondly, we consider the tangent line at the point P to the \mathbb{C} -special twisted cubic $[\mathcal{D}_k]$, we denote the intersection of this tangent line with Σ_∞ by I_{P, \mathcal{D}_k} . The conic \mathcal{D}_k has points $D_{k, \theta} = \sigma(\mathcal{C}_{k, \theta}) = (-\tau + \theta, -\tau^q + \theta, \tau\tau^q - (\tau + \tau^q)\theta + f(\theta))$. Let $g(\theta) = \tau\tau^q - (\tau + \tau^q)\theta + f(\theta)$ (the third coordinate), and multiply all coordinates by $g(\theta)^{q^2+q}$, so that the third coordinate is now in \mathbb{F}_q . Then in $\text{PG}(6, q)$, $[\mathcal{D}_k]$ has points $D_{k, \theta} = ([-(\tau + \theta)g(\theta)^q g(\theta)^{q^2}], [(-\tau^q + \theta)g(\theta)^q g(\theta)^{q^2}], g(\theta)g(\theta)^q g(\theta)^{q^2})$, for $\theta \in \mathbb{F}_q \cup \{\infty\}$. By considering $f(\theta)$, we see that $\theta = (kt_2 + 1)/k$ corresponds to the point $P = (0, 0, 1)$.

To calculate I_{P, \mathcal{D}_k} , we re-parameterise the points $D_{k, \theta}$ so that the point P of $[\mathcal{D}_k]$ has parameter ∞ . Replace the parameter $\theta \in \mathbb{F}_q \cup \{\infty\}$ by the parameter $r \in \mathbb{F}_q \cup \{\infty\}$ where $\theta = 1/r + (kt_2 + 1)/k = 1/r - w$ with w as in (3) (equivalently, $r = 1/(\theta + w)$). Using the parameter r , the twisted cubic $[\mathcal{D}_k]$ has points we denote by $D_{k, r}$, $r \in \mathbb{F}_q \cup \{\infty\}$, and we have P is the point $D_{k, \infty}$. For a fixed k , the line joining $P = D_{k, \infty}$ to $D_{k, r}$ meets Σ_∞ in a point denoted A_r , $r \in \mathbb{F}_q$. We let $r \rightarrow \infty$ to find the point $I_{P, \mathcal{D}_k} = A_\infty$. As we will only be interested in the coordinates for $D_{k, r}$ in the case when $r \rightarrow \infty$, we look at the coordinates of $D_{k, r}$ as polynomials in r , and only calculate the highest powers of r . We can write $g(\theta) = ar + \mu + \nu/r$, where $a = w^2/k - t_1w + t_0 \in \mathbb{F}_q$, and $\mu, \nu \in \mathbb{F}_{q^3}$. Then the first coordinate of $D_{k, r}$ is $-(\tau + w)a^2r^2$ plus lower powers of r . The second coordinate of $D_{k, r}$ is $-(\tau^q + w)a^2r^2$ plus lower powers of r . The third coordinate of $D_{k, r}$ is a^3r^3 plus lower powers of r . Thus $I_{P, \mathcal{D}_k} = \lim_{r \rightarrow \infty} PD_{k, r} \cap \Sigma_\infty = ([-(\tau + w)a^2], [-(\tau^q + w)a^2], 0) \equiv ([\tau], [\tau^q], 0) + w([1], [1], 0)$, as $a \in \mathbb{F}_q$. This is the same as the point I_{P, t_k} calculated above, hence the two tangent lines in $\text{PG}(6, q)$ through P are equal. \square

3.4 Consequences of the special conic representation

The results of Section 3.2 about the representation in $\text{PG}(6, q)$ of (π, ℓ_∞) -special conics of an exterior order- q -subplane π have two interesting applications. Firstly, we consider the structure of an exterior order- q -subplane π in $\text{PG}(6, q)$. We showed in [8, Theorem 4.1] that $[\pi]$ is the intersection of nine quadrics. We show in Theorem 3.11 that each of these nine quadrics contains the transversal lines of both the exterior splash \mathbb{S} and the conic cover \mathbb{C} of π . Secondly, in Theorem 3.12, we consider applications to replacement sets of $\text{PG}(5, q)$.

Theorem 3.11 *Let π be an exterior order- q -subplane of $\text{PG}(2, q^3)$ with exterior splash \mathbb{S} and conic cover \mathbb{C} . Then in the cubic extension $\text{PG}(6, q^3)$, the nine quadrics that determine $[\pi]$ each contain the transversal lines $g_{\mathbb{S}}, g_{\mathbb{S}}^q, g_{\mathbb{S}}^{q^2}$, $g_{\mathbb{C}}, g_{\mathbb{C}}^q, g_{\mathbb{C}}^{q^2}$ of \mathbb{S} and \mathbb{C} .*

Proof By Theorem 2.5, the $q^2 + q + 1$ lines of π correspond to $q^2 + q + 1$ \mathbb{S} -special twisted cubics in $[\pi]$, each lying in a 3-space through a distinct plane of \mathbb{S} . So in $\text{PG}(6, q^3)$, each of the nine quadrics defining $[\pi]$ contains $q^2 + q + 1$ points on $g_{\mathbb{S}}$. As $q^2 + q + 1 > 3$,

each of these nine quadrics contains the transversal line $g_{\mathbb{S}}$, and hence also contains its conjugates $g_{\mathbb{S}}^q, g_{\mathbb{S}}^{q^2}$.

Next we consider the $q^2 + q + 1$ (π, ℓ_{∞}) -special conics in π . By Theorem 3.5 and Corollary 3.4, these correspond to $q^2 + q + 1$ \mathbb{C} -special conics, each containing a distinct point of $g_{\mathbb{C}}$. As $q^2 + q + 1 > 3$, each of the nine quadrics containing $[\pi]$ contain $g_{\mathbb{C}}$ and similarly contain $g_{\mathbb{C}}^q$ and $g_{\mathbb{C}}^{q^2}$. \square

We conjecture that none of the nine quadrics defining $[\pi]$ contain the transversal lines $g_{\mathbb{T}}, g_{\mathbb{T}}^q, g_{\mathbb{T}}^{q^2}$ of the tangent cover \mathbb{T} , but a geometric proof eludes us.

Recall that in $\text{PG}(6, q)$, \mathcal{S} is a regular 2-spread, and the Bruck-Bose plane associated with \mathcal{S} is denoted $\mathcal{P}(\mathcal{S}) \cong \text{PG}(2, q^3)$. Let π be an exterior order- q -subplane of $\mathcal{P}(\mathcal{S})$ with exterior splash $\mathbb{S} \subset \mathcal{S}$ and conic cover \mathbb{C} . In [8], we showed that the conic cover \mathbb{C} is contained in a unique regular 2-spread (with transversal lines $g_{\mathbb{C}}, g_{\mathbb{C}}^q, g_{\mathbb{C}}^{q^2}$) denoted $\mathcal{S}_{\mathbb{C}}$. The regular 2-spread $\mathcal{S}_{\mathbb{C}}$ gives rise to the Bruck-Bose plane $\mathcal{P}(\mathcal{S}_{\mathbb{C}}) \cong \text{PG}(2, q^3)$. We consider the representation of π (and its lines and special conics) in the Bruck-Bose plane $\mathcal{P}(\mathcal{S}_{\mathbb{C}})$.

Theorem 3.12 *Let π be an exterior order- q -subplane in $\mathcal{P}(\mathcal{S}) \cong \text{PG}(2, q^3)$ with exterior splash \mathbb{S} , conic cover \mathbb{C} , and tangent cover \mathbb{T} . The points of $[\pi]$ correspond to a set of points π' in the Desarguesian plane $\mathcal{P}(\mathcal{S}_{\mathbb{C}})$ satisfying:*

1. π' is an exterior order- q -subplane of $\mathcal{P}(\mathcal{S}_{\mathbb{C}})$,
2. the lines of π correspond exactly to the (π', ℓ'_{∞}) -special conics of π' ,
3. the (π, ℓ_{∞}) -special conics of π correspond exactly to the lines of π' ,
4. π' has exterior splash \mathbb{C} , conic cover \mathbb{S} , and tangent cover \mathbb{T} .

Proof Let π be an exterior order- q -subplane in $\text{PG}(2, q^3)$ with exterior splash \mathbb{S} , conic cover \mathbb{C} and tangent cover \mathbb{T} . Label the lines of π in $\text{PG}(2, q^3)$ by $\ell_1, \dots, \ell_{q^2+q+1}$, and label the $q^2 + q + 1$ (π, ℓ_{∞}) -special conics in π by $\mathcal{C}_1, \dots, \mathcal{C}_{q^2+q+1}$. By Theorem 2.5, the line ℓ_i corresponds to an \mathcal{S} -special twisted cubic $[\ell_i]$ in a 3-space about a plane of \mathbb{S} . By Theorem 3.5, the (π, ℓ_{∞}) -special conic \mathcal{C}_i corresponds to a \mathbb{C} -special twisted cubic in a 3-space about a plane of \mathbb{C} . By Theorem 2.5, in the Desarguesian plane $\mathcal{P}(\mathcal{S}_{\mathbb{C}})$, π' is a set of $q^2 + q + 1$ points, with lines $\mathcal{C}_1, \dots, \mathcal{C}_{q^2+q+1}$. As the points and (π, ℓ_{∞}) -special conics of π form a Desarguesian plane, π' is an exterior order- q -subplane of $\mathcal{P}(\mathcal{S}_{\mathbb{C}})$. Further by Theorem 3.6, π' has (π', ℓ'_{∞}) -special conics $\ell_1, \dots, \ell_{q^2+q+1}$. Moreover in $\text{PG}(6, q)$, π' has exterior splash \mathbb{C} , conic cover \mathbb{S} , and tangent cover \mathbb{T} . \square

We relate this theorem to the nine quadrics that define $[\pi]$ discussed above. In $\text{PG}(6, q)$, the sets $[\pi]$, $[\pi']$ of Theorem 3.12 are equal, so the nine quadrics used to define $[\pi]$ are

the same as the nine quadrics used to define $[\pi']$. This is consistent with Theorem 3.11 which shows that in $\text{PG}(6, q^3)$, the nine quadrics defining $[\pi]$ all contain $g_{\mathbb{S}}, g_{\mathbb{S}}^q, g_{\mathbb{S}}^{q^2}$ and $g_{\mathbb{C}}, g_{\mathbb{C}}^q, g_{\mathbb{C}}^{q^2}$, and the roles of \mathbb{S} and \mathbb{C} are reversed in $[\pi]$ and $[\pi']$ by Theorem 3.12.

In a similar way to Theorem 3.12, we can consider the unique 2-spread $\mathcal{S}_{\mathbb{T}}$ (with transversal lines $g_{\mathbb{T}}, g_{\mathbb{T}}^q, g_{\mathbb{T}}^{q^2}$) containing the tangent cover \mathbb{T} of π . In the Bruck-Bose plane $\mathcal{P}(\mathcal{S}_{\mathbb{T}})$, consider the set of points π'' corresponding to the points of $[\pi]$. Then π'' is a set of $q^2 + q + 1$ points. To define ‘lines’ of π'' , we consider the pointsets in π'' that correspond to \mathbb{T} -special twisted cubics. By Corollary 3.9, these ‘lines’ have at most three points. So π'' is not an order- q -subplane of $\mathcal{P}(\mathcal{S}_{\mathbb{T}})$.

4 Order- q -subplanes with a common splash

The article [7] briefly looked at order- q -subplanes that share a common splash. In particular, [7, Theorem 7.2] showed that two order- q -subplanes with a common splash either share a common order- q -subline, or meet in at most three points. Further (see Theorem 2.3) there are exactly two order- q -subplanes with a common exterior splash and a common order- q -subline. In this section, we further the investigation of exterior order- q -subplanes with a common splash.

Let π_1, π_2 be the two exterior order- q -subplanes with a common exterior splash \mathbb{S} and common order- q -subline b . In Section 4.1, we investigate the two families of order- q -sublines in \mathbb{S} in relation to π_1 and π_2 . Further, we show that if π_1 has conic cover \mathbb{C} and tangent cover \mathbb{T} , then π_2 has conic cover \mathbb{T} and tangent cover \mathbb{C} . In Section 4.2, we give a relationship between the (π_1, ℓ_{∞}) -special conics of π_1 and the (π_2, ℓ_{∞}) -special conics of π_2 . In Section 4.3, we start with an exterior splash \mathbb{S} and appropriate order- q -subline b , and give a geometric construction in the $\text{PG}(6, q)$ setting of the two order- q -subplanes that share \mathbb{S} and b . Note that this is analogous to the construction in [6] which gives a similar construction in the case of an order- q -subplane tangent to ℓ_{∞} with a fixed tangent splash on ℓ_{∞} .

4.1 Order- q -sublines in a common splash

By [19], an exterior splash \mathbb{S} of ℓ_{∞} in $\text{PG}(2, q^3)$ contains $2(q^2 + q + 1)$ order- q -sublines that lie in two distinct *families* of size $q^2 + q + 1$. If π is an order- q -subplane with exterior splash \mathbb{S} , then the two families of \mathbb{S} are characterised in relation to π in [7, Theorem 5.2]. If P is a point of π , then the lines of π through P meet \mathbb{S} in an order- q -subline called a π -*pencil-subline* of \mathbb{S} . The other family of order- q -sublines of \mathbb{S} can be constructed from ‘special’ dual conics in π and these order- q -sublines are called π -*dual-conic-sublines* of \mathbb{S} . Note also that by [7, Theorem 5.3], there is some other exterior order- q -subplane π' with common

exterior splash \mathbb{S} for which the π -pencil-sublines of \mathbb{S} are the π' -dual-conic-sublines of \mathbb{S} . That is, the classification of the two families *depends* on the associated order- q -subplane. In this section we improve this result by finding the relationship between the two order- q -subplanes π, π' . We show that the two order- q -subplanes with common exterior splash \mathbb{S} and sharing an order- q -subline b (which exist by Theorem 2.3) interchange the roles of the two families of order- q -sublines of \mathbb{S} .

Theorem 4.1 *Let π_1, π_2 be the two exterior order- q -subplanes of $\text{PG}(2, q^3)$ with common exterior splash \mathbb{S} and a common order- q -subline b . Then the π_1 -pencil-sublines of \mathbb{S} are the π_2 -dual-conic-sublines of \mathbb{S} (and the π_1 -dual-conic-sublines of \mathbb{S} are the π_2 -pencil-sublines of \mathbb{S}).*

Proof By Theorem 2.1, we can without loss of generality assume one of the order- q -subplanes is \mathcal{B} . Recall that in $\text{PG}(2, q^3)$, we construct \mathcal{B} from $\pi_0 = \text{PG}(2, q)$ by applying the homography of Theorem 2.2 with matrix K . We construct a second order- q -subplane using the homography ψ of $\text{PG}(2, q^3)$ with matrix $K' = HK$, where

$$H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and so} \quad K' = \begin{pmatrix} -\tau^q & 1 & 0 \\ -\tau & 1 & 0 \\ \tau\tau^q & -\tau - \tau^q & 1 \end{pmatrix}.$$

The collineation ψ maps π_0 to an order- q -subplane π . Further \mathcal{B} and π have the same exterior splash \mathbb{S} on ℓ_∞ . Also note that both \mathcal{B} and π share the order- q -subline $\{K(0, y, z)^t : y, z \in \mathbb{F}_q\}$. Finally the point with coordinates $K(1, 0, 0)^t = (-\tau, -\tau^q, \tau\tau^q)^t$ lies in \mathcal{B} but not in π , so $\mathcal{B} \neq \pi$.

Note that the homography ζ with matrix H interchanges \mathcal{B} and π . Further, by the proof of [7, Theorem 5.3], ζ fixes \mathbb{S} and interchanges the two families of order- q -sublines of \mathbb{S} . Hence the \mathcal{B} -pencil-sublines of \mathbb{S} are the π -dual-conic-sublines of \mathbb{S} , and conversely. \square

We now show that if π_1, π_2 are the two distinct exterior order- q -subplanes with common exterior splash and common order- q -subline, then the roles of the conic cover and tangent cover are interchanged for π_1 and π_2 .

Theorem 4.2 *Let π_1, π_2 be the two exterior order- q -subplanes of $\text{PG}(2, q^3)$ with a common exterior splash \mathbb{S} and containing a common order- q -subline b . Then the conic cover of π_1 is the tangent cover of π_2 (and conversely, the tangent cover of π_1 is the conic cover of π_2).*

Proof By Theorem 2.1, we can without loss of generality prove this for the two order- q -subplanes \mathcal{B}, π used in the proof of Theorem 4.1. Label the conic cover and tangent cover of \mathcal{B} by $\mathbb{C}_{\mathcal{B}}$ and $\mathbb{T}_{\mathcal{B}}$ respectively, and the conic cover and tangent cover of π by \mathbb{C}_{π} and \mathbb{T}_{π} respectively.

By [8, Theorem 2.2], we can without loss of generality prove the result for the point $P = (0, 0, 1)$, which lies in the common order- q -subline $\{K(0, y, z)^t : y, z \in \mathbb{F}_q\}$ of \mathcal{B} and π . The tangent plane \mathcal{T}_P at $P = (0, 0, 1)$ is defined in [8, Theorem 4.2], it is a plane in $\text{PG}(6, q)$ through P that meets Σ_∞ in a line $\ell = \mathcal{T}_P \cap \Sigma_\infty$ lying in a cover plane α of the tangent cover $\mathbb{T}_\mathcal{B}$ of \mathcal{B} . By [8, Theorem 7.2], the line ℓ lies in a unique 2-regulus of \mathbb{S} which corresponds to an order- q -subline d of $\mathbb{S} \subset \ell_\infty$ in $\text{PG}(2, q^3)$. Further, d is the \mathcal{B} -pencil-subline of \mathbb{S} arising from the intersection of the lines of \mathcal{B} through P with ℓ_∞ .

Now consider the homography ζ with matrix H defined in the proof of Theorem 4.1. As ζ interchanges the two families of order- q -sublines of \mathbb{S} , $d' = \zeta(d)$ is a \mathcal{B} -dual-conic-subline of \mathbb{S} . Hence by [8, Theorem 7.2], in $\text{PG}(6, q)$, $[\zeta]$ maps the cover plane α of $\mathbb{T}_\mathcal{B}$ to a cover plane β of $\mathbb{C}_\mathcal{B}$. Further, by Theorem 4.1, $d' = \zeta(d)$ is a π -pencil-subline of \mathbb{S} . Note that ζ fixes P , so d' is the order- q -subline arising from the intersection of the lines of π through P with ℓ_∞ . Hence by [8, Theorem 7.2], β is a plane of the tangent cover \mathbb{T}_π of π . That is, $\mathbb{C}_\mathcal{B} = \mathbb{T}_\pi$, as required. \square

4.2 Special conics of the two subplanes

Let π_1, π_2 be the two exterior order- q -subplanes of $\text{PG}(2, q^3)$ with common exterior splash \mathbb{S} , and a common order- q -subline b (recall these exist by Theorem 2.3). There is no obvious relationship between the order- q -sublines of π_1 and the order- q -sublines of π_2 , although we can pair them up by matching order- q -sublines from each subplane that share a common point of \mathbb{S} . However, there is a nice geometric relationship between certain special conics of π_1 and π_2 . In particular, there are $q + 1$ conics in $\text{PG}(2, q^3)$ that meet both π_1 and π_2 in special conics.

Theorem 4.3 *Let π_1, π_2 be the two exterior order- q -subplanes with common exterior splash and a common order- q -subline b . Let \mathcal{C} be a (π_1, ℓ_∞) -special conic of π_1 that is tangent to b . Then the extension of \mathcal{C} to $\text{PG}(2, q^3)$ meets π_2 in a (π_2, ℓ_∞) -special conic.*

Proof Let π_1, π_2 be two exterior order- q -subplanes with common exterior splash \mathbb{S} and order- q -subline b (these exist by Theorem 2.3). Let \mathbb{S} have carriers E_1, E_2 , and third conjugate point E_3 . For each point $P_i \in b$, $i = 1, \dots, q + 1$, there is a unique conic \mathcal{C}_i of $\text{PG}(2, q^3)$ that contains E_1, E_2, E_3 and P_i , and is tangent to b at the point P_i . The conic \mathcal{C}_i meets the order- q -subplane π_1 (respectively π_2) in a conic that contains P_i , is tangent to b at P_i , and contains E_1, E_2, E_3 (in the extension to $\text{PG}(2, q^3)$). Hence for $i = 1, \dots, q + 1$, $\mathcal{C}_i \cap \pi_1$ is a (π_1, ℓ_∞) -special conic of π_1 , and $\mathcal{C}_i \cap \pi_2$ is a (π_2, ℓ_∞) -special conic of π_2 . \square

4.3 Construction of two subplanes in $\text{PG}(6, q)$

In $\text{PG}(2, q^3)$, let \mathbb{S} be an exterior splash of ℓ_∞ , and let b be an exterior order- q -subline whose extension to $\text{PG}(2, q^3)$ meets \mathbb{S} . By Theorem 2.3, there are exactly two distinct exterior order- q -subplanes π_1, π_2 with exterior splash \mathbb{S} containing b . We now give a geometric construction in the $\text{PG}(6, q)$ setting of these two order- q -subplanes π_1, π_2 . Note that by Theorem 4.2, one of the covers of \mathbb{S} acts as the conic cover for π_1 , and the other cover of \mathbb{S} acts as the conic cover for π_2 . The next construction uses each set of cover planes of \mathbb{S} to construct one of the order- q -subplanes containing b with exterior splash \mathbb{S} .

Construction 1 In $\text{PG}(2, q^3)$, $q \geq 3$, let \mathbb{S} be an exterior splash of ℓ_∞ , let ℓ be a line through a point L of \mathbb{S} , and let $b = \{P_1, \dots, P_{q+1}\}$ be an order- q -subline of ℓ that is exterior to ℓ_∞ . Let \mathbb{X}, \mathbb{Y} be the two covers of \mathbb{S} . In $\text{PG}(6, q)$, for $i, j \in \{1, \dots, q+1\}$, let

1. $L_{ij} = P_i P_j \cap [L]$, $i \neq j$,
2. $L_{ii} = m_i \cap [L]$, where m_i is the unique tangent to the \mathbb{S} -special twisted cubic $[b]$ at the point P_i ,
3. α_{ij} be the unique cover plane of the cover \mathbb{X} through the point L_{ij} ,
 β_{ij} be the unique cover plane of the cover \mathbb{Y} through the point L_{ij} ,
4. $\Sigma_{ij} = \langle \alpha_{ij}, P_i \rangle$, $\Gamma_{ij} = \langle \beta_{ij}, P_i \rangle$.

Then the two exterior order- q -subplanes π_1, π_2 with common exterior splash \mathbb{S} and order- q -subline b are constructed as follows.

- I. $[\pi_1]$ consists of the points $\Sigma_{ij} \cap \Sigma_{kl}$ for $i, j, k, l \in \{1, \dots, q+1\}$, $\{i, j\} \neq \{k, l\}$.
- II. $[\pi_2]$ consists of the points $\Gamma_{ij} \cap \Gamma_{kl}$ for $i, j, k, l \in \{1, \dots, q+1\}$, $\{i, j\} \neq \{k, l\}$.

Proof Let π_1, π_2 be the two order- q -subplanes with exterior splash \mathbb{S} containing b (which exist by Theorem 2.3). By Theorem 4.2, one of the covers, \mathbb{X} say, acts as the conic cover for π_1 , and the other cover \mathbb{Y} acts as the conic cover for π_2 . We prove the construction for $[\pi_1]$, and $[\pi_2]$ is similar. Note that the $q+1 + \binom{q+1}{2}$ points L_{ij} , $1 \leq i \leq j \leq q+1$, are distinct by [17, 21.1.9]. Hence the $q+1 + \binom{q+1}{2}$ 3-spaces Σ_{ij} , $1 \leq i \leq j \leq q+1$, are distinct.

We first show that each 3-space Σ_{ij} meets $[\pi_1]$ in a set of points that corresponds to a (π_1, ℓ_∞) -special conic. Fix i , and let $j \in \{1, \dots, q+1\}$, and firstly suppose $j \neq i$. In $\text{PG}(2, q^3)$, the two points P_i, P_j of b lie in a unique (π_1, ℓ_∞) -special conic of π_1 , denoted \mathcal{C}_{ij} . In $\text{PG}(6, q)$, $[\mathcal{C}_{ij}]$ is an \mathbb{X} -special twisted cubic in a 3-space Π_{ij} that contains the

line $P_i P_j$, and a plane of \mathbb{X} (the conic cover of π_1), so Π_{ij} contains the plane α_{ij} , that is, $\Pi_{ij} = \Sigma_{ij}$. Now consider the case $j = i$. For each point $P_i \in b \subset \text{PG}(2, q^3)$, there is a unique (π_1, ℓ_∞) -special conic \mathcal{C}_i of π_1 that is tangent to b at the point P_i . In $\text{PG}(6, q)$, $[\mathcal{C}_i]$ is an \mathbb{X} -special twisted cubic in a 3-space Π_i about a plane of \mathbb{X} . By Theorem 3.10, the line $P_i L_{ii}$ is in the 3-space Π_i , hence $\Pi_i = \Sigma_{ii}$. Hence for each i, j , Σ_{ij} meets $[\pi_1]$ in a set of points that corresponds to a (π_1, ℓ_∞) -special conic which contains one or two points of b .

We now show that every point in π_1 lies on at least two (π_1, ℓ_∞) -special conics that meet b . This is clearly true for each point $P_i \in b$. Let P be a point of $\pi_1 \setminus b$. For each point $P_i \in b$, there is a unique (π_1, ℓ_∞) -special conic through P and P_i . These special conics are not necessarily distinct, but there are at least $(q+1)/2$ distinct ones. As $q \geq 3$, the point P lies on at least two (π_1, ℓ_∞) -special conics that meet b . Hence in $\text{PG}(6, q)$, the point P lies in at least two of the 3-spaces Σ_{ij} . Note that the Σ_{ij} pairwise meet in a point, hence by determining all the intersections $\Sigma_{ij} \cap \Sigma_{kl}$, we construct all the points of π_1 . \square

We note that there does not appear to be a corresponding construction in $\text{PG}(2, q^3)$, as it is difficult to distinguish between the points of π_1 and π_2 . With the aim of producing a simpler construction of the exterior order- q -subplane, we very briefly discuss another way of characterising the order- q -subplanes in $\text{PG}(6, q)$. This time we work in the quadratic extension $\text{PG}(6, q^2)$ of $\text{PG}(6, q)$, so that we can use the *chords* of a twisted cubic (see [17]). For a point $P \in \text{PG}(6, q^2)$, let P^σ denote the conjugate of P arising from the Frobenius automorphism $\sigma(x) = x^q$ in $\text{GF}(q^2)$. Let \mathcal{N} be a twisted cubic in a 3-space Σ . There are three types of chords of \mathcal{N} in Σ : real chords PQ for $P, Q \in \mathcal{N}$, $P \neq Q$; imaginary chords PP^σ for $P \in \mathcal{N} \setminus \mathcal{N}$; and one tangent chord for each $P \in \mathcal{N}$. Note that every point of Σ lies on a unique chord of \mathcal{N} , see [17].

Let b be an exterior order- q -subline of $\text{PG}(2, q^3)$, so by Theorem 2.5, $\mathcal{N} = [b]$ is an \mathcal{S} -special twisted cubic in a 3-space about a plane $\alpha \in \mathcal{S}$. We can uniquely label the points of the plane α using the $q^2 + q + 1$ pairs denoted (P, Q) , (P, P^σ) , (P, b) as follows. A point $R \in \alpha$ lies on a unique chord of \mathcal{N} . We label R by the pair: (P, Q) if R lies on the real chord PQ of \mathcal{N} ; (P, P^σ) if R lies on the imaginary chord PP^σ of \mathcal{N} ; and (P, b) if R lies on the tangent b to \mathcal{N} through P . Hence if π is an order- q -subplane of $\text{PG}(2, q^3)$ with exterior splash \mathbb{S} , then we have a unique labelling in $\text{PG}(6, q)$ for each point in each plane of \mathbb{S} .

We consider the interpretation of this pair labelling in planes of the covers \mathbb{C} and \mathbb{T} of π . The interpretation in planes of the conic cover \mathbb{C} is straightforward: Let $\beta \in \mathbb{C}$, then by the results of Section 3.2, there is a unique (π, ℓ_∞) -special conic \mathcal{C} in π corresponding to β . The points in β are labelled by pairs as follows. 1. Pairs (P, Q) for $P, Q \in \mathcal{C}$. 2. Pairs (P, P^σ) for points $P \in \mathcal{C}^* \setminus \mathcal{C}$. 3. Pairs (P, ℓ) for $P \in \mathcal{C}$ and ℓ a line of π through P . Note that Theorem 3.10 ensures that the labelling in 3 works. The interpretation in planes of the tangent cover \mathbb{T} is more difficult, and we leave this as an open question.

5 Intersection of two exterior splashes

In this section we investigate the intersection of two exterior splashes. First recall from Section 4.1 that an exterior splash \mathbb{S} of ℓ_∞ in $\text{PG}(2, q^3)$ contains $2(q^2 + q + 1)$ order- q -sublines that lie in two distinct *families* of size $q^2 + q + 1$. Note that two distinct order- q -sublines in the same family meet in exactly one point, and order- q -sublines in different families meet in 0, 1 or 2 points.

Ferret and Storme [16, Lemma 2.3] showed that two exterior splashes meet in at most $2q + 2$ points. Lavrauw and Van de Voorde [19, Remark 24] show that this bound is tight by exhibiting an example of two exterior splashes whose intersection has size $2q + 2$, this intersection consists of two disjoint order- q -sublines, one from each family.

In this section, we show in Theorem 5.2 that two exterior splashes cannot meet in two order- q -sublines of the same family. Then in Theorem 5.3, we characterise the maximal intersection of two exterior splashes by showing that if the intersection has size $2q + 2$, then it consists of two disjoint order- q -sublines, one from each family.

We need the next lemma which follows from [8, Theorem 7.2].

Lemma 5.1 *Let \mathcal{R} be a 2-regulus contained in an exterior splash \mathbb{S} .*

1. *Each of the $q^2 + q + 1$ ruling lines of \mathcal{R} lie in a unique cover plane of \mathbb{S} , further these $q^2 + q + 1$ cover planes are in the same cover of \mathbb{S} .*
2. *Each line in a cover plane of \mathbb{S} is the ruling line for one 2-regulus contained in \mathbb{S} . Further, the $q^2 + q + 1$ 2-reguli constructed from the lines in one cover plane correspond to the $q^2 + q + 1$ order- q -sublines in one family of \mathbb{S} .*
3. *Let ℓ, m be ruling lines for distinct 2-reguli contained in \mathbb{S} . If ℓ and m meet, then $\langle \ell, m \rangle$ is a cover plane of \mathbb{S} .*

Theorem 5.2 *Let \mathbb{S} be an exterior splash of ℓ_∞ in $\text{PG}(2, q^3)$, and let b_1, b_2 be order- q -sublines of \mathbb{S} that belong to the same family. If \mathbb{S}' is any exterior splash containing b_1 and b_2 , then $\mathbb{S}' = \mathbb{S}$.*

Proof Let b_1 and b_2 be distinct order- q -sublines of \mathbb{S} in the same family, so b_1 and b_2 meet in a unique point P . By Theorem 2.5, in $\text{PG}(6, q)$, $[b_1], [b_2]$ are 2-reguli contained in the regular 2-spread \mathcal{S} . Let Q be any point of the spread plane $[P]$, then there is a ruling line t of $[b_1]$ through Q , and a ruling line u of $[b_2]$ through Q . By Lemma 5.1, $\langle t, u \rangle$ is a cover plane of \mathbb{S} . Hence the planes of the regular 2-spread \mathcal{S} that meet $\langle t, u \rangle$ are exactly the planes of \mathbb{S} , that is, \mathbb{S} is completely determined. \square

Theorem 5.3 *If two exterior splashes of ℓ_∞ in $\text{PG}(2, q^3)$ meet in $2q + 2$ points, then these points correspond to two disjoint order- q -sublines, one from each family.*

Proof We work in $\text{PG}(6, q)$, so let $\mathbb{S}_1, \mathbb{S}_2$ be two exterior splashes of the regular 2-spread \mathcal{S} , and suppose that $\mathbb{S}_1, \mathbb{S}_2$ meet in $2q + 2$ planes. By Theorem 2.5, we need to show that $\mathbb{S}_1 \cap \mathbb{S}_2$ consists of two disjoint 2-reguli. We first show that $\mathbb{S}_1 \cap \mathbb{S}_2$ contains a 2-regulus. Let γ be a plane in $\mathbb{S}_1 \cap \mathbb{S}_2$ and P a point in γ . Let α be a cover plane of \mathbb{S}_1 through P , and let β be a cover plane of \mathbb{S}_2 through P . We consider two cases.

1. Suppose $\alpha \cap \beta$ is a line. By Lemma 5.1, each line of α (respectively β) is a ruling line of a distinct 2-regulus of \mathcal{S} , and this 2-regulus is contained in \mathbb{S}_1 (respectively \mathbb{S}_2). So the line $\alpha \cap \beta$ is the ruling line of a 2-regulus contained in $\mathbb{S}_1 \cap \mathbb{S}_2$, as required.
2. Suppose that α, β do not meet in a line, so $\alpha \cap \beta = P$. We consider two sub-cases.
 - (a) Firstly suppose some line ℓ of α through P meets at least three planes of $(\mathbb{S}_1 \cap \mathbb{S}_2) \setminus \{\gamma\}$. By Lemma 5.1, the line ℓ is a ruling line of a unique 2-regulus $\mathcal{R} \subset \mathbb{S}_1$. Hence the 3-space $\Pi = \langle \ell, \beta \rangle$ meets at least three planes of $(\mathbb{S}_1 \cap \mathbb{S}_2) \setminus \{\gamma\}$ in lines. Hence by [19, Lemma 7], Π meets all the planes in \mathcal{R} in lines. Thus β meets all the planes in \mathcal{R} , and so \mathcal{R} also lies in \mathbb{S}_2 . So in this case, $\mathbb{S}_1 \cap \mathbb{S}_2$ contains the 2-regulus \mathcal{R} .
 - (b) For the second case, we suppose that each line of α through P meets at most two of the planes in $(\mathbb{S}_1 \cap \mathbb{S}_2) \setminus \{\gamma\}$. As $|(\mathbb{S}_1 \cap \mathbb{S}_2) \setminus \{\gamma\}| = 2q + 1$, q of the lines in α through P meet two planes of $(\mathbb{S}_1 \cap \mathbb{S}_2) \setminus \{\gamma\}$, and the remaining line of α through P meets one plane. A counting argument shows that the 4-space $\langle \alpha, \beta \rangle$ contains exactly one plane δ of the regular 2-spread \mathcal{S} , and meets the remaining q^3 planes of \mathcal{S} in a line. That is, α, β, δ all lie in the 4-space $\langle \alpha, \beta \rangle$, δ is a plane of \mathcal{S} , and α, β are cover planes of an exterior splash contained in \mathcal{S} . Hence it follows that δ meets both α and β in points. As $\delta \cap \alpha$ is a point, $\delta \in \mathbb{S}_1$. Similarly, $\delta \in \mathbb{S}_2$, so $\delta \in \mathbb{S}_1 \cap \mathbb{S}_2$.

We now show that the point $P = \alpha \cap \beta$ lies in δ . Suppose not, that is, $P \notin \delta$. Now one line of α through P contains the point $\delta \cap \alpha$, and (by the assumption above) q lines of α through P meet two planes of $(\mathbb{S}_1 \cap \mathbb{S}_2) \setminus \{\gamma\}$. So let ℓ be a line of α through P , not through $\delta \cap \alpha$, that meets two planes γ_1, γ_2 of $(\mathbb{S}_1 \cap \mathbb{S}_2) \setminus \{\gamma\}$ in the points G_1, G_2 respectively. Hence the 3-space $\Pi = \langle \ell, \beta \rangle$ meets γ_1 and γ_2 in lines. Further, both δ and Π lie in the 4-space $\langle \alpha, \beta \rangle$, so Π meets δ in a line. That is, Π meets three planes $\gamma_1, \gamma_2, \delta$ of $(\mathbb{S}_1 \cap \mathbb{S}_2) \setminus \{\gamma\}$ in lines. These three lines lie in a unique 1-regulus \mathcal{R}' of Π . So $\Pi \cap \alpha$ contains the three non-collinear points G_1, G_2 , and $\delta \cap \alpha$. Hence $\alpha \subset \Pi$, contradicting $\alpha \cap \beta$ being exactly the point P . Hence the point P lies in δ , and so $\delta = \gamma$. So Π meets the three planes $\gamma_1, \gamma_2, \gamma$ of $\mathbb{S}_1 \cap \mathbb{S}_2$ in lines, and these three lines lie in a unique 1-regulus \mathcal{R}' of Π . So \mathcal{R}' lies in the 2-regulus \mathcal{R} defined by $\gamma_1, \gamma_2, \gamma$. As ℓ meets each line

of \mathcal{R}' , ℓ is a ruling line of \mathcal{R} and so $\mathcal{R} \subset \mathbb{S}_1$. Moreover, as $\mathcal{R}' = \mathcal{R} \cap \Pi$, β meets each plane of \mathcal{R} , and so $\mathcal{R} \subset \mathbb{S}_2$. That is, \mathcal{R} is a 2-regulus in $\mathbb{S}_1 \cap \mathbb{S}_2$.

So we have shown that $\mathbb{S}_1 \cap \mathbb{S}_2$ contains a 2-regulus \mathcal{R} . Let ℓ be a ruling line of the 2-regulus \mathcal{R} . Then by Lemma 5.1, there is a unique cover plane α_1 of \mathbb{S}_1 through ℓ , and a unique cover plane α_2 of \mathbb{S}_2 through ℓ . So α_1 meets $\mathbb{S}_1 \cap \mathbb{S}_2 \setminus \mathcal{R}$ in $q + 1$ points of $\alpha_1 \setminus \ell$, and similarly for α_2 . So the 3-space $\langle \alpha_1, \alpha_2 \rangle$ meets $\mathbb{S}_1 \cap \mathbb{S}_2 \setminus \mathcal{R}$ in $q + 1$ lines. By [19, Lemma 10], these $q + 1$ lines form a 1-regulus we denote \mathcal{R}_1 . Let m be a transversal line of \mathcal{R}_1 . Then m meets $q + 1$ planes of $\mathbb{S}_1 \cap \mathbb{S}_2 \setminus \mathcal{R}$. Now m is the ruling line of a unique 2-regulus \mathcal{R}' which is contained in both \mathbb{S}_1 and \mathbb{S}_2 , hence $\mathcal{R}' = \mathbb{S}_1 \cap \mathbb{S}_2 \setminus \mathcal{R}$. So in $\text{PG}(6, q)$, $\mathbb{S}_1 \cap \mathbb{S}_2$ contains two disjoint 2-reguli $\mathcal{R}, \mathcal{R}'$. Hence, by Theorem 2.5, in $\text{PG}(2, q^3)$, $\mathbb{S}_1 \cap \mathbb{S}_2$ contains two disjoint order- q -sublines. As the order- q -sublines are disjoint, they belong to different families. \square

Corollary 5.4 *Let $\mathbb{S}_1, \mathbb{S}_2$ be two exterior splashes of $\text{PG}(6, q)$ that meet in $2q + 2$ planes. Then each cover plane in the cover \mathbb{X} of \mathbb{S}_i meets $\mathbb{S}_1 \cap \mathbb{S}_2$ in a set of $2q + 2$ points that form a line and an \mathbb{X} -special conic.*

Proof We continue from the end of the proof of Theorem 5.3, using the same notation as in the last paragraph of the proof. In particular, the 3-space denoted $\langle \alpha_1, \alpha_2 \rangle$ meets $\mathbb{S}_1 \cap \mathbb{S}_2$ in a 1-regulus \mathcal{R}_1 ; and \mathcal{R}_1 meets $\alpha_1 \setminus \ell$ in $q + 1$ points. As a 1-regulus meets a plane in either two lines, or in an irreducible conic, \mathcal{R}_1 meets α_1 in an irreducible conic \mathcal{C} exterior to ℓ . The points of \mathcal{C} lie in $q + 1$ distinct planes of $\mathbb{S}_1 \cap \mathbb{S}_2$, namely the planes of the 2-regulus denoted \mathcal{R}' above. Hence by [8, Theorems 7.2, 7.3] \mathcal{C} is an \mathbb{X} -special conic, where \mathbb{X} is the cover of \mathbb{S} that contains α_1 . So $\mathbb{S}_1 \cap \mathbb{S}_2$ meets α_1 in the line ℓ and an \mathbb{X} -special conic. A similar result holds for the plane α_2 . Moreover, this situation generalises to any cover plane of \mathbb{S}_i . \square

6 Conclusion

This article completes our investigation of exterior order- q -subplanes of $\text{PG}(2, q^3)$ begun in [7, 8]. We note that the proofs in this article are a mix of geometric arguments and coordinate based arguments. It would be nice to exploit the underlying geometry of the structure to prove more results using a geometric argument.

A recurrent theme of the results on exterior order- q -subplanes is that the conics and twisted cubics of greatest interest are “special”. To summarise, we have for a fixed exterior splash \mathbb{S} with conic cover \mathbb{C} and tangent cover \mathbb{T} :

- \mathbb{C} -special conics in a plane of \mathbb{C} (and \mathbb{T} -special conics in a plane of \mathbb{T}) correspond to order- q -sublines of \mathbb{S} (in different families).

- \mathbb{S} -special twisted cubics correspond to lines of an associated order- q -subplane.
- \mathbb{C} -special twisted cubics (and \mathbb{T} -special twisted cubics) correspond to special conics in some associated order- q -subplane.

Each result has been proved individually, however, it would be interesting to have a unifying theory to explain these relationships. We note that Theorem 3.11 (where we show that the nine quadrics defining $[\pi]$ in $\text{PG}(6, q)$ all contain the transversals $g_{\mathbb{S}}$ and $g_{\mathbb{C}}$) goes some way towards understanding this.

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